Section 11.3: The Integral Test and Estimates of Sums

Objective: In this lesson, you learn

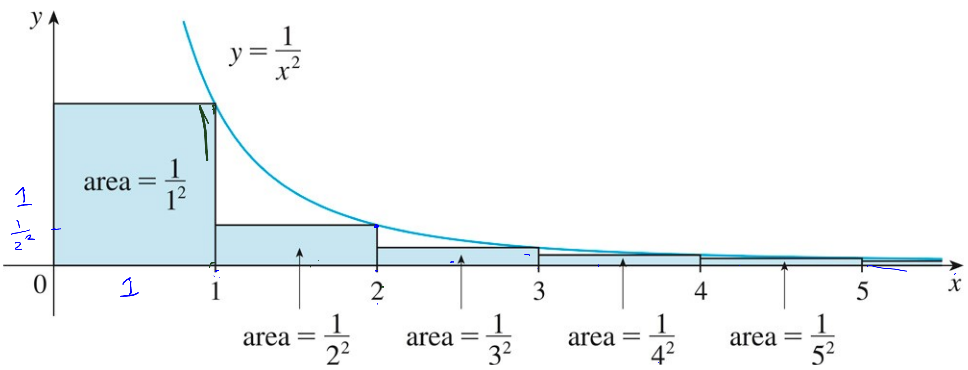
- ☐ how to develop the Integral Test to determine whether or not a series is convergent or divergent without explicitly finding its sum.

Problem: Compare

$$\underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{p=2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Recall that: $\int_1^\infty \frac{1}{x^p}\,dx$ = convergent if $p > 1$

is a convergent Integral.



$$\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) \le 1 + \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}_{\text{convergent}} \quad = 1$$

→ From page 47 'improper integral.'

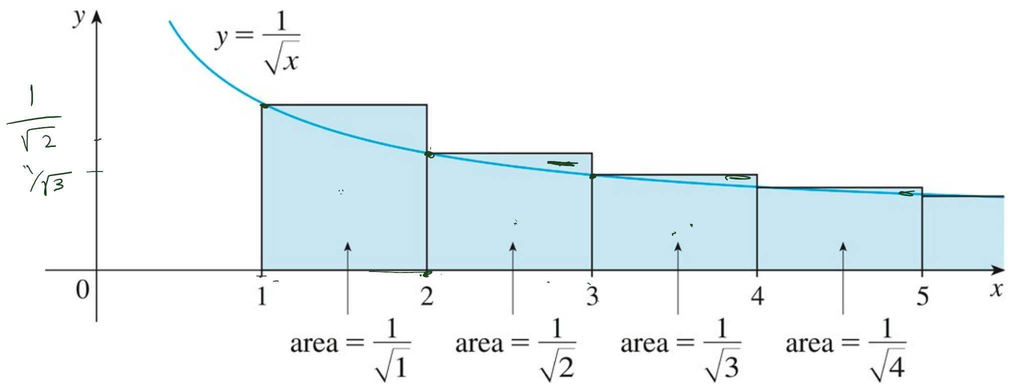
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$$

⇒ which means $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Problem: Compare

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

divergent
$p = 1/2$



$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \geq \int_1^\infty \frac{1}{\sqrt{x}}\,dx \to \text{divergent}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent series.

P- Integral

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{conv.} & p > 1 \\ \text{div} & p \leq 1 \end{cases}$$

p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv.} & p > 1 \\ \text{div.} & p \leq 1 \end{cases}$$

I. The Integral Test

Except for **geometric** series and the **telescoping** series, it is difficult to find the exact sum of a series. So we try to determine the convergence of a series without explicitly finding the sum.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[a, \infty)$ and let $a_n = f(n)$. Then the series

$$\sum_{n=a}^{\infty} a_n$$

is **convergent** if and only if the **improper integral**

$$\int_a^{\infty} f(x)\, dx$$

is covergent. That is,

a. If $\int_a^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=a}^{\infty} a_n$ is convergent.

b. If $\int_a^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=a}^{\infty} a_n$ is divergent.

Note that: In general, $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\, dx$.

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1 \quad , \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\int_1^{\infty} \frac{1}{x^4}\, dx = \frac{1}{3} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Example 1: For what values of p is the $p-$**series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent} & p > 1. \\ \text{div.} & p \leq 1. \end{cases}$$

Example 2: Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$$

form TD :

$$\lim_{n\to\infty} \frac{n^2}{1+n^2} = 1 \neq 0$$

$$\sum_{n=1}^{\infty} \frac{n^2}{1+n^2} \text{ is divergent.}$$

Example 3: Does the series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

TD: $\lim_{n\to\infty} \frac{1}{1+n^2} = \frac{1}{\infty} = 0$ test failed (try another test).

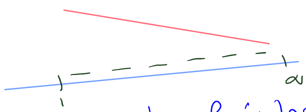
Integral test: let $f(x) = \frac{1 > 0}{1+x^2 > 0}$, $[1,\infty)$

1. f is conts on $\mathbb{R}$ → conts on $[1,\infty)$

$$\left(\frac{h}{g}\right)' = \frac{g h' - h \cdot g'}{g^2}$$

2. $f(x)$ is positive → $[1,\infty)$

3. $f'(x) = \frac{-1 \times 2x}{(1+x^2)^2}$ < 0 ; $2x > 0$, $(1+x^2)^2 > 0$



↳ f is decreasing on $(1,\infty)$

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_1^{\infty} = \lim_{\ell\to\infty} \tan^{-1}(x)\Big|_1^{\ell}$$

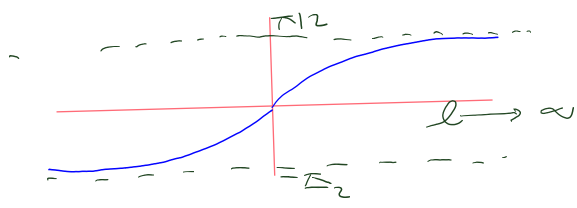
$$= \lim_{\ell\to\infty} \tan^{-1}(\ell) - \tan^{-1}(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\int_1^{\infty} \frac{1}{1+x^2}\,dx$ is conv. then $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ is conv. by integral test.

or $1+x^2 > x^2$ → $\frac{1}{1+x^2} \le \frac{1}{x^2}$ → $\int_1^{\infty} \frac{1}{1+x^2}\,dx \le \int_1^{\infty} \frac{1}{x^2}\,dx$

↳ convergent by comparison; convergent by $p = 2$



$\ln \infty = \infty$

Example 4: Test the series

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} = \sum_{n=2} \frac{(\ln n)^2}{n} = 0 + \frac{(\ln 2)^2}{2} + \frac{(\ln 3)^2}{3} + \cdots$$

for convergence or divergence.

TD: $\lim_{n\to\infty} \frac{(\ln n)^2}{n} \to \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{2(\ln n) * \frac{1}{n}}{1}$

$= \lim_{n\to\infty} 2\frac{\ln n}{n} \to \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} \longrightarrow 0$ test failed

Integral test: $f(x) = \frac{(\ln x)^2 > 0}{x > 0}$, $x \geq 1$

$\frac{1}{0}$, $\ln \leq 0$, $\sqrt{-\text{neg}}$.

1. f is conts on $[1, \infty)$
2. f is positive.
3. $f'(x) = \frac{\cancel{x}\cdot 2(\ln x)\cdot \frac{1}{\cancel{x}} - (\ln x)^2}{x^2} = \frac{2\ln x - [\ln(x)]^2}{x^2} = \frac{\ln x\,(2 - \ln(x)) = 0}{x^2 \neq 0}$

$f'(x)$ on $(1, \infty)$

$2 - \ln x = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$

$1 \quad + + + + \quad e^2 \quad - - - - - - - \quad \infty$

f is ↗↘ on $[e^2, \infty)$

So, $\int_{e^2}^{\infty} f(x)\,dx = \int_{e^2}^{\infty} \frac{(\ln x)^2}{x}\,dx$ $\qquad u = \ln x \quad du = \frac{1}{x}dx$, $x = e^2 \Rightarrow u = 2$, $x \to \infty \Rightarrow u \to \infty$

$= \int_2^{\infty} u^2\,du = \frac{u^3}{3}\Big|_2^{\infty} = \lim_{\ell\to\infty} u^3\Big|_2^{\ell} = \lim_{\ell\to\infty} u^3 - 8 = \infty$

So, $\int_{e^2}^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ is divergent by integral test.

$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} = \sum_{n=1}^{7} \frac{(\ln n)^2}{n} + \sum_{n=8}^{\infty} \frac{(\ln(n))^2}{n} \to$ divergent.